Desingularization in boundary element analysis of three-dimensional Stokes flow

Maximilian Harmel*, Michał P. Rajski and Roger A. Sauer

Aachen Institute for Advanced Study in Computational Engineering Science (AICES), RWTH Aachen University, Templergraben 55, 52056 Aachen, Germany

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Abstract: The boundary element method (BEM) is able to solve partial differential equations without volumetric discretization and integration. Therefore, the BEM is able to reduce the compuational as well as the meshing effort compared to volumetric methods like classical finite elements. In this work, a conventional and a nonsingular BEM formulation for Stokes flow are presented and investigated in three-dimensions, considering rotating spheres within a viscous fluid.

1 Introduction

The boundary element method (BEM) is able to solve partial differential equations (PDEs) efficiently. To this end, the solution of a PDE is expressed in terms of boundary distributions of their fundamental solution. Thus, boundary discretization and integration is sufficient to solve a domain problem. The computational effort is drastically reduced compared to volumetric methods such as the classical finite element method (FEM). The BEM is capable to solve several linear PDEs (e.g. Laplace, Helmholtz or biharmonic equation). In this work the focus is on accurate numerical integration for Stokes flow problems. Therefore, a nonsingular boundary element formulation is introduced and compared to the conventional one.

2 Boundary element method (BEM) for Stokes Flow

In Stokes flow (Reynolds number $Re \ll 1$) convective inertial forces are negligibly small compared to viscous forces. Further assuming small accelerations, the motion of an incompressible Newtonian fluid in an external domain $\mathcal{D}$ with boundary $\mathcal{S}$ can be described by the steady Stokes equation

$$-\nabla p + \eta\,\nabla^2 \boldsymbol{v} = \boldsymbol{0} \quad \text{in}\,\mathcal{D}, \tag{1}$$

in combination with properly chosen boundary conditions. For fixed domains in absence of normal fluid velocity ($v_n = 0$), the incompressibilty constraint is automatically accounted. Pressure and velocity of the fluid are denoted by p and $\boldsymbol{v}$ respectively, while η denotes its viscosity and ρ its mass density. Equation (1) can be transformed into the boundary integral equation (BIE)

$$c(\boldsymbol{y})\,v_j(\boldsymbol{y}) = -\frac{1}{8\pi\eta}\int_{\mathcal{S}} G_{ij}(\boldsymbol{r})\,t_i(\boldsymbol{x})\,\mathrm{d}a_x + \frac{1}{8\pi}\int_{\mathcal{S}} v_i(\boldsymbol{x})\,T_{ijk}(\boldsymbol{r})\,n_k(\boldsymbol{x})\,\mathrm{d}a_x\;, \quad \forall\,\boldsymbol{y}\in\mathcal{S}\;, \tag{2}$$

*corresponding author, email: harmel@aices.rwth-aachen.de
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see e.g. Pozrikidis (2002). The outward unit normal is denoted by $\boldsymbol{n}$, while $\boldsymbol{t}$ denotes the boundary traction $\boldsymbol{t} = \boldsymbol{\sigma}\,\boldsymbol{n}$ with stress tensor $\boldsymbol{\sigma}$. The fundamental solution for velocity and traction can be obtained by

$$G_{ij}(\boldsymbol{r}) = \frac{\delta_{ij} + \bar{r}_i \bar{r}_j}{r}, \qquad T_{ijk}(\boldsymbol{r}) = -6\,\frac{\bar{r}_i \bar{r}_j \bar{r}_k}{r^2}\ , \tag{3}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, $r = \|\boldsymbol{r}\|$ and $\bar{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$. From (3) it can be seen that the fundamental solutions become singular as $\boldsymbol{x}$ approaches $\boldsymbol{y}$. Therefore special care should be taken while integrating. A general overview of avaliable techniques for singular integration can be found in Huang and Cruse (1993).
The BIE is spatially discretized by n_n isogeometric basis functions. Those provide desirable continuity properties, that are beneficial for the accuracy of the BEM. Using collocation points $\boldsymbol{y}_\alpha$, where $\alpha \in 1, \ldots, n_n$, the BIE (2) can be written in compact form (see e.g. Heltai et al. (2017)) as

$$(\mathbf{I} - \mathbf{C})\,\mathbf{M}\,\mathbf{v} = \boldsymbol{\mathcal{G}}\,\mathbf{t} + \boldsymbol{\mathcal{T}}\,\mathbf{v}\ , \tag{4}$$

where $\boldsymbol{\mathcal{G}}$ contains the contributions from the first integral in (2), while $\boldsymbol{\mathcal{T}}$ contains the contributions from the second one. Operator $\mathbf{M}$ transforms the nodal values to the collocation points. Fig. 1 illustrates the choice of collocation and quadrature points on a spherical isogeometric surface. The parametric coordinates of the collocation points are given by the Greville abscissae (Greville, 1964). Moreover, the elements are divided into four parts such that all collocation points lie at the boundaries of those sub-elements. Thus, it is guaranteed that the quadrature point at position $\boldsymbol{x}$ does not coincide with any collocation point $\boldsymbol{y}_\alpha$.

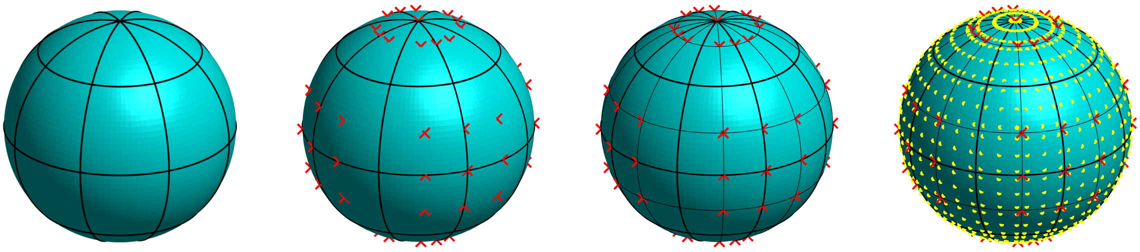

Figure 1: Boundary quadrature (from left to right): 32 isogeometric elements, 62 collocation points $\boldsymbol{y}_\alpha$ (red) chosen with the Greville abscissae, sub-elements for quadrature, position of quadrature points (yellow, here: 3×3 Gaussian quadrature points per sub-element 1152 quadrature points in total)

The nonsingular boundary element method was introduced by Klaseboer et al. (2009, 2012) and then extended for isogeometric collocation by Heltai et al. (2014). Here, this approach is investigated and compared to the conventional BEM. In the nonsingular BEM known solutions are superimposed to unknown fields in such a way that singularities are removed completely. Following Heltai et al. (2014) the desingularized system of equations can be written as

$$\mathbf{H}\mathbf{u} = \mathbf{L}\mathbf{f}\ , \tag{5}$$

where $\mathbf{H} := \boldsymbol{\mathcal{G}} + \mathbf{C}\mathbf{M}$ and $\mathbf{L} := \boldsymbol{\mathcal{T}} + \mathbf{B}\mathbf{M}$, while $\mathbf{C}$ and $\mathbf{B}$ contain information about the known solution. Operators $\mathbf{C}$ and $\mathbf{B}$ correct the singular entries on the right hand side of (4).
The conventional and the nonsingular BEM for Stokes flow is investigated in numerical experiments. A sphere that rotates within a viscous fluid is considered here. Fig. 2 shows the norm of the resulting surface traction error

$$\mathrm{e}_t(\boldsymbol{x}) = \frac{\|\boldsymbol{t}\,(\boldsymbol{x})\| - \|\boldsymbol{t}^*\,(\boldsymbol{x})\|}{\max\|\boldsymbol{t}^*\|}\ , \tag{6}$$

where $\boldsymbol{t}^*$ is the analytical solution of the traction. The number of quadrature points per sub-element is denoted by n_{qp}, while m denotes the mesh refinement level (e.g. $m = 2$ in Fig. 1).

With increasing mesh and quadrature refinement, the computational error is decreasing for both BEM formulations. However, the nonsingular BEM yields a drastically reduced error compared to the conventional one (see Fig. 2, right: error reduction by more than 80% for $m = 2$).

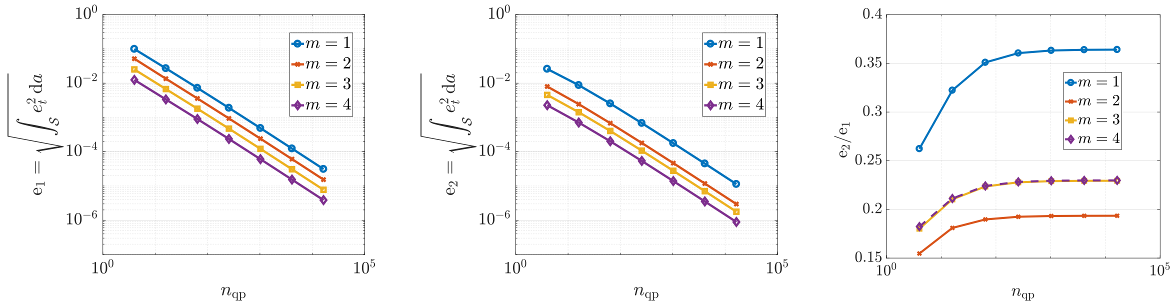

Figure 2: Rotating sphere in a viscous fluid (from left to right): Resulting numerical error e_1 for conventional BEM, e_2 for nonsingular BEM and their relation e_1/e_2. The number of quadrature points per element is denoted by n_{qp}, while m denotes the mesh refinement level.

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